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Nonlinear Wave Equations, Covariant Exterior Derivative, Painlevé Test, and Integrability

W.-H. Steeb,[1] J. A. Louw,[1] and A. Bringer[2]

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A class of nonlinear wave equations is derived which can be written as the vanishing of a covariant exterior derivative. The Painlevé test is performed and the connection with integrability is discussed.

Nonlinear evolution equations are usually called integrable when one of the following properties is fulfilled: (I) the initial value problem can be solved exactly with the help of the inverse scattering transform, (II) they have an infinite number of conservation laws, (III) they have an auto-Bäcklund transformation or a Bäcklund transformation, (IV) besides Lie point vector fields, they admit Lie Bäcklund vector fields, (V) they describe pseudospherical surfaces, i.e., surfaces of constant negative Gaussian curvature, or (VI) they can be written as covariant exterior derivatives of Lie algebra-valued differential forms. It is conjectured that if property I holds, then the properties II–VI also hold.

The connection between integrable systems and covariant exterior derivatives has been studied by various authors (Crampin *et al.*, 1977; Crampin, 1978; Sasaki, 1979a–c; Nestrenko, 1980, 1981; Madore, 1983; Steeb *et al.*, 1984a).

In the present paper we give a class of nonlinear wave equations in one space dimension which can be written as covariant exterior derivatives. The connection with other properties of integrable systems mentioned above will be described. Moreover, the Painlevé property of this class of wave equations will be studied.

[1]Department of Physics, Rand Afrikaans University, Johannesburg 2000, Republic of South Africa.
[2]Institut fur Festkörperforschung, KFA Jülich, D-5170 Jülich, West Germany.

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The wave equation in one space dimension under consideration is given by

$$u_{xx} - u_{xx} = f(u) \tag{1}$$

where f is a smooth function. Shadwick (1978) showed that equation (1) admits Bäcklund transformations if the function f satisfies the linear ordinary differential equation

$$f'' + cf = 0 \qquad (c \in R) \tag{2}$$

Equation (1) together with equation (2) includes the Liouville equation and the sine Gordon equation. Steeb (1984a) showed that equation (1) admits Lie Bäcklund vector fields if f satisfies equation (2). Furthermore, it can be shown that equation (1) together with equation (2) passes the so-called Painlevé test for partial differential equations (Weiss *et al.*, 1983) when we perform a suitable transformation for the dependent variable u. For example, for the Liouville equation $u_{tt} - u_{xx} = e^u$ we put $v = \exp(u)$ (Steeb *et al.*, 1984b). For a critical discussion of the Painlevé test we refer to Steeb and Louw (1986).

Let us now introduce the covariant exterior derivative. Let M be a C^∞ finite-dimensional manifold (dim $M = n$). Denote by $F^p(M)$ the set of all C^∞ p-differential forms on M for each $p = 0, 1, \ldots, m$. Let L be a finite-dimensional Lie algebra. A Lie algebra-valued differential form on M is an element of the tensor product $F^p(M) \otimes L$. If T_i, $i = 1, \ldots, n$, is a basis for L, then a Lie-algebra valued p-differential form $\tilde{\gamma}$ can be written as

$$\tilde{\gamma} = \sum_{i=1}^{n} \gamma_i \otimes T_i \tag{3}$$

where $\gamma_i \in F^p(M)$. The bracket [,] of Lie algebra-valued differential forms $\tilde{\gamma}$ and $\tilde{\beta}$ is defined as

$$[\tilde{\gamma}, \tilde{\beta}] := \sum_{i=1}^{n} \sum_{j=1}^{n} (\gamma_i \wedge \beta_j) \otimes [T_i, T_j] \tag{4}$$

where $[T_i, T_j]$ denotes the commutator of the Lie algebra L.

The exterior derivative of a Lie algebra-valued differential form is given by

$$d\tilde{\gamma} = \sum_{i=1}^{n} (d\gamma_i) \otimes T_i \tag{5}$$

The covariant exterior derivative of a Lie algebra-valued p-form $\tilde{\beta}$ with respect to a Lie algebra one-form $\tilde{\alpha}$ is defined as

$$D_{\tilde{\alpha}}\tilde{\beta} := d\tilde{\beta} - g[\tilde{\alpha}, \tilde{\beta}] \tag{6}$$

where

$$g = \begin{cases} -1 & p \text{ even} \\ -1/2 & p \text{ odd} \end{cases} \tag{7}$$

Consequently,

$$D_{\tilde{\alpha}}\tilde{\alpha} = d\tilde{\alpha} + \tfrac{1}{2}[\tilde{\alpha}, \tilde{\alpha}] \tag{8}$$

The Lie algebra-valued one-form $\tilde{\alpha}$ is called the connection.

To study the wave equation (1) we consider the case where $M = \mathbb{R}^2$ and $L = sl(2, \mathbb{R})$. Extensions will be given at the end of the paper. In local coordinates (x, t) a Lie algebra-valued one-differential-form is given by

$$\tilde{\alpha} = \sum_{i=1}^{3} \alpha_i \otimes T_i \tag{9}$$

where

$$\alpha_i = a_i\, dx + A_i\, dt \tag{10}$$

and $\{T_1, T_2, T_3\}$ is a basis of the Lie algebra $sl(2, \mathbb{R})$. A convenient choice is

$$T_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad T_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad T_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{11}$$

The condition that the covariant derivative vanishes

$$D_{\tilde{\alpha}}\tilde{\alpha} = 0 \tag{12}$$

leads to the following system of PDEs of first order:

$$-\partial a_1/\partial t + \partial A_1/\partial x + a_2 A_3 - a_3 A_2 = 0 \tag{13a}$$

$$-\partial a_2/\partial t + \partial A_2/\partial x + 2(a_1 A_2 - a_2 A_1) = 0 \tag{13b}$$

$$\partial a_3/\partial t + \partial A_3/\partial x - 2(a_1 A_3 - a_3 A_1) = 0 \tag{13c}$$

The sine Gordon equation

$$u_{tt} - u_{xx} + \sin u = 0 \tag{14}$$

can be represented as follows:

$$a_2 = -\tfrac{1}{4}(\cos u + 1), \qquad A_1 = \tfrac{1}{4}(\cos u - 1)$$
$$a_2 = \tfrac{1}{4}(u_x + u_t - \sin u), \qquad A_2 = \tfrac{1}{4}(u_x + u_t + \sin u) \tag{15}$$
$$a_3 = -\tfrac{1}{4}(u_x + u_t + \sin u), \qquad A_3 = -\tfrac{1}{4}(u_x + u_t - \sin u)$$

If we insert a_1 through A_3 into equation (13), then we find that equation (13a) is satisfied identically. Equations (13b) and (13c) yield the sine-Gordon equation. Motivated by this, we find the following extension.

Theorem. Let

$$a_1 = f_1(u), \qquad A_1 = f_2(u)$$
$$a_2 = c_1 u_x + c_2 u_t + f_3(u), \qquad A_2 = c_3 u_x + c_4 u_t + f_4(u) \tag{16}$$
$$a_3 = c_5 u_x + c_6 u_t + f_5(u), \qquad A_3 = c_7 u_x + c_8 u_t + f_6(u)$$

where $f_1, \ldots, f_6$ are smooth functions and $c_1, \ldots, c_8 \in \mathbb{R}$. Then the Lie algebra-valued differential form $\tilde{\alpha}$ satisfies the condition (12) if

(i) $c_1 = c_2 = c_3 = c_4, \qquad c_5 = c_6 = c_7 = c_8$ (17a)

(ii) $f_4 = -f_3, \qquad f_6 = -f_5$ (17b)

(iii) $f_5 = c f_3 \, (c \in \{+1, -1\})$ (17c)

(iv) $-c_1 u_{tt} + c_1 u_{xx} + 2f_3(-f_1 - f_2) = 0$ (17d)

and for $c = 1$

(va) $\dot{f}_1 = -4c_1 f_3, \qquad \dot{f}_2 = 4c_1 f_3, \qquad \ddot{f}_3 = -16c_1^2 f_3$ (17e)

where $c_1 = -c_5$; for $c = -1$

(vb) $\dot{f}_1 = 4c_1 f_3, \qquad \dot{f}_2 = -4c_1 f_3, \qquad \ddot{f}_3 = 16c_1^2 f_3$ (17f)

where $c_1 = c_5$.

Let us now discuss the solutions to equation (17). We find at once the nonlinear wave equations

$$u_{tt} - u_{xx} = C_1 \cosh u + C_2 \sinh u \tag{18}$$

and

$$u_{tt} - u_{xx} = C_1 \sin u + C_2 \cos u \tag{19}$$

($C_1, C_2 \in \mathbb{R}$) can be written as the covariant exterior derivative of a Lie algebra-valued one-form, where the underlying Lie algebra is $sl(2, \mathbb{R})$.

The nonlinear wave equation

$$u_{tt} - u_{xx} = e^{u} - e^{-2u} \tag{20}$$

does not belong to the class given above. To include such types of equations, we have to extend the Lie algebra.

Let us now discuss whether or not these equations pass the so-called Painlevé test for partial differential equations. Ward (1984) has introduced what is now called the Painlevé property for partial differential equations. The system of partial differential equations is considered in the complex domain. Let n be the number of independent variables. Assume that the system of partial differential equations has coefficients that are analytic on $\mathbb{C}^n$. The Painlevé property is defined as follows: if S is an analytic noncharacteristic complex hypersurface in $\mathbb{C}^n$, then every solution of the partial differential equations that is analytic on $\mathbb{C}^n \backslash S$ is meromorphic on $\mathbb{C}^n$.

To prove whether or not a given partial differential equation has the Painlevé property is very difficult, or even impossible. A weaker form (so-called Painlevé test) of the Painlevé property was proposed by Weiss *et al.* (1983). They looked for solutions of the form

$$u = \Phi^m \sum_{j=0}^{\infty} u_j \Phi^j \tag{21}$$

where Φ is an analytic function whose vanishing defines a noncharacteristic hypersurface S. Inserting this expansion into the partial differential equation leads to conditions on m and recursion relations for the functions u_j. The weaker form of the Painlevé property states that m should be an integer, that the recursion relation should be consistent, and that the series expansion should contain the correct number of arbitrary functions. We say that the partial differential equation passes the Painlevé test. Notice that it may happen that more than one branch arises. Moreover, the expansion could *a priori* miss some essential singularities.

The precise connection between the Painlevé property and integrability or solvability remains mysterious. For ordinary differential equations some results are available (Yoshida, 1983a, b). Now two conjectures can be made: (I) Assume that the partial differential equation is integrable. Then it has the Painlevé property. (II) Assume that the partial differential equation has the Painlevé property. Then it is integrable. Conjecture I is not true. Examples are the Harry Dym equation $u_t = u^3 u_{xxx}$ and the diffusion equation $u_t = (u^{-2} u_x)_x$. Both equations fail the Painlevé test. It seems that conjecture II is true. The problem is that, while the Painlevé test can easily be performed, it requires a great deal of work to show that essential singularities do not occur. Even in most cases it is not possible to perform this calculation. On the other hand, in almost all practical cases we only need to perform the

Painlevé test. An exception is the equation $u_t^2 = 2uu_x^2 - (1+u^2)u_{xx}$, which passes the test, but does not have the Painlevé property (Clarkson, 1985a).

Let us now study the wave equation

$$u_{tt} - \sum_{i=1}^{n} u_{x_i x_i} = f(u) \tag{22}$$

where f is a smooth function of u. Let $n = 1$ and let us assume that f satisfies the ordinary differential equation $f'' + cf = 0$, where $c = 1$ or $c = -1$. Then equation (22) is completely integrable. Moreover, with these assumptions it can be shown that the equation passes the Painlevé test, where for the case $c = 1$ we have to introduce $v = \exp(iu)$ and for the case $c = -1$ we have to introduce $v = \exp(u)$. For two and more space dimensions and with the same assumptions for f equation (22) does not pass the Painlevé test. For $f(u) = au + bu^3$ we find that for every space dimension equation (22) does not pass the test.

Let us now discuss what group-theoretic reductions tell us (Steeb *et al.*, 1985; Clarkson, 1985b). Let $n = 3$. Then equation (22) admits the symmetry generators

$$\begin{gathered} \partial/\partial t,\ \partial/\partial x_1,\ \partial/\partial x_2,\ \partial/\partial x_3,\ x_1\,\partial/\partial t + t\,\partial/\partial x_1,\ x_2\,\partial/\partial t + t\,\partial/\partial x_2 \\ x_3\,\partial/\partial t + t\,\partial/\partial x_3,\ x_1\,\partial/\partial x_2 - x_2\,\partial/\partial x_1 \\ x_2\,\partial/\partial x_3 - x_3\,\partial/\partial x_2,\ x_3\,\partial/\partial x_1 - x_1\,\partial/\partial x_3 \end{gathered} \tag{23}$$

The knowledge of the symmetry generators enables us to introduce a similarity variable s and a similarity ansatz. The similarity ansatz yields an ordinary differential equation. Then for this equation we can test whether or not the ordinary differential equation passes the Painlevé test.

Consider first $n = 1$. The symmetry generator $x\,\partial/\partial t + t\,\partial/\partial x$ leads to the symmetry variable $s = t^2 - x^2$ and the similarity ansatz $u(x, t) = g(s)$. It follows that

$$g'' + g'/s = f(g)/4s \tag{24}$$

where $g' = dg/ds$. Assume that f satisfies $f'' + cf = 0$, where $c = 1$ or $c = -1$ and $f(u) \neq 0$. Then, together with the transformation $w = \exp(ig)$ $(c = 1)$ or $w = \exp(g)$ $(c = -1)$, we find the third Painlevé transcendent, which, of course, passes the Painlevé test. The reduction with the generators $\partial/\partial t$ and $\partial/\partial x$ also leads to a node that has the Painlevé property.

Consider now the case with two space dimensions. From the symmetry generators $x_1\,\partial/\partial t + t\,\partial/\partial x_1$ and $x_2\,\partial/\partial t + t\,\partial/\partial x_2$ we obtain the similarity ansatz $u(x_1, x_2, t) = g(s)$, where $s = t^2 - x_1^2 - x_2^2$. It follows that

$$g'' + \tfrac{3}{2}g' = f(g)/4s \tag{25}$$

Assume that f satisfies $f''+cf=0$ ($c=1$ or $c=-1$) and $f(u)\neq 0$. Then, together with the transformation $w=\exp(ig)$ ($c=1$) or $w=\exp(g)$ ($c=-1$), we find a node for which we can perform the Painlevé test. We find that the Painlevé test is not passed. On the other hand, with the ansatz $u(x_1, x_2, t)=g(s)$, where $s=k_1x_1+k_2x_2-\omega t$, we find that

$$(\omega^2-k_1^2-k_2^2)g''=f(g) \tag{26}$$

where $\omega^2<k_1^2+k_2^2$. Here we find that the Painlevé property is fulfilled.

With this in mind we can discuss the result of Leibrandt (1978), who found a Bäcklund transformation between the sine-Gordon equation in two space dimensions and the sinh-Gordon equation in two space dimensions. Due to the results given above we conjecture that this Bäcklund transformation only links plane traveling waves. Notice that Hirota (1976) found a three-soliton solution by the nonlinear superposition of plane traveling waves. We also refer to the discussion of this point by Weiss (1984).

Let us now study wave equations that do not have the Painlevé property. Consider the sine-Gordon equation in two space dimensions,

$$\partial^2u/\partial t^2-\partial^2u/\partial x^2-\partial^2u/\partial y^2+\sin u=0 \tag{27}$$

When we introduce polar coordinates (r, Φ) and omit the dependence on Φ, equation (27) takes the form

$$\partial^2u/\partial t^2-\partial^2u/\partial r^2-(1/r)\,\partial u/\partial r+\sin u=0 \tag{28}$$

Performing the transformation $v=\exp(iu)$, we arrive at

$$v(\partial^2v/\partial t^2-\partial^2v/\partial r^2)-(\partial v/\partial t)^2+(\partial v/\partial r)^2$$
$$-(1/r)v\,\partial v/\partial r+\tfrac{1}{2}(v^3-v)=0 \tag{29}$$

The Painlevé test yields

$$v_0=-4(\Phi_t^2-\Phi_r^2), \qquad v_1=4[\Phi_{tt}-\Phi_{rr}-(1/r)\Phi_r] \tag{30}$$

At the resonance $r=2$ we obtain

$$0=(16/r)\Phi_r(\Phi_{tt}\Phi_r^2+\Phi_r\Phi_t^2-2\Phi_{rt}\Phi_r\Phi_t) \tag{31}$$

If Φ depends only on r, then the right-hand side of equation (31) vanishes. The right-hand side of equation (31) also vanishes if

$$\Phi_r^2\Phi_{tt}+\Phi_t^2\Phi_{rr}-2\Phi_t\Phi_r\Phi_{rr}=0 \tag{32}$$

Equation (32) admits the symmetry generators

$$\partial/\partial t,\ \partial/\partial r,\ r\,\partial/\partial t+t\,\partial/\partial r,\ r\,\partial/\partial r+t\,\partial/\partial t+\Phi\,\partial/\partial\Phi \tag{33}$$

The group-theoretic reduction with the help of the ansatz $\Phi(r, t)=f(s)$ ($s=r^2-t^2$) yields

$$sf'^3=0 \tag{34}$$

Any plane wave $\Phi(r, t) = f(kr - \omega t)$ is a solution to equation (32). No dispersion relation arises, i.e., k and ω can be chosen arbitrarily. In addition, equation (32) is invariant under the Möbius group

$$\Phi = (a\psi + b)/(c\psi + d) \tag{35}$$

where $ad - bc = 1$. Equation (35) can be viewed as a particular auto-Bäcklund transformation. The inverse transformation is given by

$$\psi = (d\psi - b)/(-c\psi + a) \tag{36}$$

Notice that equation (32) is even invariant under $\Phi = F(\psi)$, where F is a twice-differentiable function. Equation (32) has also been discussed by Weiss (1984) in connection with the double sine-Gordon equation. Equation (32) is closely related to the Born–Infeld equation in one space dimension

$$(1 + u_x^2)u_{tt} + (-1 + u_t^2)u_{xx} - 2u_t u_x u_{xt} = 0 \tag{37}$$

Any plane wave $u(x, t) = f(kx - \omega t)$ is a solution to equation (37) where $\omega^2 = k^2$ (dispersion relation). Equation (37) can be viewed as a sum of the linear wave equation $u_{tt} - u_{xx} = 0$ and equation (32) ($\Phi \to u$). Thus, the dispersion relation is generated by the linear wave equation. The Born–Infeld equation (37) can be derived from the Lagrangian

$$L = -(1 + u_x^2 - u_t^2)^{1/2} \tag{38}$$

Barbashov and Chernikov (1966) solved the initial value problem. However, the solution can only be given in parametric form. Equation (37) is of the hyperbolic type if $1 + u_x^2 - u_t^2 > 0$. When we put $y = it$ we find

$$(1 + u_y^2)u_{xx} + (1 + u_x^2)u_{yy} - 2u_x u_y u_{xy} = 0 \tag{39}$$

This equation is always of the elliptic type.

Let us apply the Painlevé test to equation (37). Inserting the ansatz

$$u \sim \Phi^n u_0 \tag{40}$$

we find two branches, namely (i) $n = 1$ and (ii) n arbitrary. In both cases u_0 can be chosen arbitrarily. Thus we say that equation (37) passes the Painlevé test.

Consider now the Liouville equation in two space dimensions

$$u_{tt} - u_{xx} - u_{yy} = \exp(u) \tag{41}$$

Putting $v = \exp(u)$, we obtain

$$v(v_{tt} - v_{xx} - v_{yy}) - v_t^2 + v_x^2 + v_y^2 - v^3 = 0 \tag{42}$$

Performing the Painlevé test, we find that

$$j=0: \qquad v_0=2(\Phi_t^2-\Phi_x^2-\Phi_y^2) \tag{43}$$

$$j=1: \qquad v_1=-2(\Phi_{tt}-\Phi_{xx}-\Phi_{yy}) \tag{44}$$

At the resonance $r=2$ we obtain

$$\begin{aligned}0&=\Phi_t^2(\Phi_{xx}\Phi_{yy}-\Phi_{xy}^2)+\Phi_x^2(\Phi_{tt}\Phi_{yy}-\Phi_{yt}^2)+\Phi_y^2(\Phi_{tt}\Phi_{xx}-\Phi_{xt}^2)\\&\quad+2\Phi_x\Phi_y(\Phi_{xt}\Phi_{yt}-\Phi_{xy}\Phi_{tt})+2\Phi_x\Phi_t(\Phi_{ty}\Phi_{xy}-\Phi_{xt}\Phi_{yy})\\&\quad+2\Phi_t\Phi_y(\Phi_{xt}\Phi_{xy}-\Phi_{yt}\Phi_{xx})\end{aligned} \tag{45}$$

Any plane wave $\Phi(x, y, t)=f(k_1x+k_2x-\omega t)$ is a solution to equation (45). No dispersion relation arises. Equation (45) admits the symmetry generators

$$\begin{gathered}\partial/\partial t, \partial/\partial x, \partial/\partial y, x\,\partial/\partial y-y\,\partial/\partial x\\ x\,\partial/\partial t+t\,\partial/\partial x, y\,\partial/\partial t+t\,\partial/\partial y\end{gathered} \tag{46}$$

The group-theoretic reduction $\Phi(x, y, t)=f(t^2-x^2-y^2)$ yields

$$sf'^4=0 \tag{47}$$

Equation (45) is invariant under the Möbius group (35).

Consider now the wave equation

$$u_{tt}-u_{xx}=au+bu^3 \tag{48}$$

and group-theoretic reductions. Let $u(x, t)=g(s)$ and $s=t^2-x^2$. Then we obtain

$$g''+g'=ag/4s+bg^3/4s \tag{49}$$

We find that this equation does not have the Painlevé property. Let us now study wave equations of the form

$$u_{tt}-u_{xx}=P(u)/Q(u) \tag{50}$$

where P and Q are polynomials in u. We assume that $Q(u)\neq 0$ for all u, $P\neq Q$, and the degree of the polynomials P and Q is higher than one. An example is given by

$$u_{tt}-u_{xx}=au+bu^3/(1+u^2) \tag{51}$$

where $a, b\in\mathbb{R}$. This equation can be derived from the Lagrangian

$$L(u, u_t, u_x)=u_t^2-u_x^2-\ln(1+u^2) \tag{52}$$

We ask: Is there any P and Q such that equation (50) has the Painlevé property? The group-theoretic reduction $u(x, t)=g(s)(s=t^2-x^2)$ yields

$$Q(g)g''+Q(g)g'=P(g)/4s \tag{53}$$

With the assumptions made for Q and P we find that equation (53) does not have the Painlevé property. Here we have used the list given by Davis (1962). Consequently, equation (50) does not have the Painlevé property.

Finally, let us study the wave equation (48) and the Painlevé test due to Weiss *et al.* (1983). For the sake of simplicity we put $a=1$ and $b=-1$. Inserting the ansatz $u \sim \Phi^n u_0$ into equation (48), we obtain $n=-1$ and $u_0 \neq 0$. The dominant terms are u_{tt}, u_{xx}, and u^3. The resonances are given by $r_1=-1$ and $r_2=4$. Inserting the expansion

$$u=\Phi^{-1} \sum_{j=0}^{\infty} u_j \Phi^j \tag{54}$$

into equation (20), we find that

$$j=0: \quad u_0^2=2(\Phi_x^2-\Phi_t^2) \tag{55}$$

$$j=1: \quad 3u_0^2 u_1=(\Phi_{tt}-\Phi_{xx})u_0+2(\Phi_t u_{0t}-\Phi_x u_{0x}) \tag{56}$$

$$j=2: \quad 3u_0^2 u_2=u_{0xx}-u_{0tt}-3u_0 u_1^2+u_0 \tag{57}$$

$$\begin{aligned} j=3: \quad 2u_0^2 u_3 &= (\Phi_{xx}-\Phi_{tt})u_2-6u_0 u_1 u_2 \\ &\quad -2(\Phi_t u_{2t}-\Phi_x u_{2x})-u_1^3+u_1+u_{1xx}-u_{1tt} \end{aligned} \tag{58}$$

At the resonance $j=4$ we obtain

$$\begin{aligned} 0 &= 2(\Phi_{tt}-\Phi_{xx})u_3+6u_0 u_1 u_3+4(\Phi_t u_{3t}-\Phi_x u_{3x})+u_{2tt}-u_{2xx} \\ &\quad -u_2+3u_0 u_2^2+3u_1^2 u_2 \end{aligned} \tag{59}$$

Inserting equations (55)-(58) into equation (59), we find that the right-hand side does not vanish identically. Consequently, equation (20) does not have the Painlevé property. This coincides with the fact that the group-theoretic reduction with the help of the ansatz $u(x, t)=g(s)$ $(s=t^2-x^2)$ leads to an ordinary differential equation that does not have the Painlevé property. This related to the fact that $\Phi(x, t)=t^2-x^2$ does not satisfy equation (59). When we insert $\Phi(x, t)=f(kx-t)$ into equation (59), we find that the right-hand side is identically zero.

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